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On the moments of a (WS)$^\beta$ distribution

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Abstract. The introduction of local optical potential wells of a generalised Wood-Saxon (WS) type in optical-model studies, and also in some developments of the energy density formalism, is necessary to deal with the moments of a Wood-Saxon distribution raised to a (in general non-integral) real power (WS)$^\beta$.

In this paper, the analytic structure of these moments $\mathscr{D}_{n,\beta}(a, R)$ is analysed. In fact, an expansion is derived for them that for intermediate and large values of $x = R/a$ reduces to a polynomial of order n in x.

At the same time, rapidly converging series expansions are obtained in order to represent the coefficients appearing in this polynomial and, in addition, a recursion relation that makes their numerical computation easier is established.

Finally, by assuming a (WS)$^\beta$ parametrisation, this polynomial is applied to the study of the A dependence of several quantities of relevance in the geometrical characterisation of density distributions and optical potential wells corresponding to medium-mass and heavy nuclei.

1. Introduction

Much effort in the past has been devoted to the study and interpretation of the scattering of 20-50 MeV α particles from medium-mass nuclei [1-4].

At forward angles this scattering process has been described with the optical model using complex weakly energy-dependent Wood-Saxon potentials. At intermediate and large angles, it turns out that the differential cross section is larger than might be expected by simple extrapolation of the results obtained at forward angles. This 'anomalous large-angle scattering' (ALAS) (see references cited in [4]) could not be suitably explained with the optical model using conventional Wood-Saxon potentials.

In order to surmount these difficulties within the optical model, in addition to folding potentials [5-7], phenomenological local spherical potentials of the form

$$U(r) = -V_0 f_0(r) - \mathrm{i} V_1 f_1(r) + V_c(r) \tag{1.1}$$

(where $V_c(r)$ stands for the nuclear Coulomb potential) have been introduced, whose real and imaginary parts have been parametrised with form factors of type

$$f_i(r) = \{1 + \exp[(r - R_i)/a_i]\}^{-\beta} \qquad i = 0, 1 \tag{1.2}$$

where β may assume integral and non-integral values [8] (see also [4] and references therein). In another type of modification, the real part of the potentials is represented by means of a *spline function* [9] or expanded in part (or in whole) by means of spherical Bessel functions [10].

When $\beta = 1$ in equation (1.2), the half-value radius $R_{1/2}$ and the surface thickness t, defined as $W_i(R_{1/2}) = \frac{1}{2} W_i(0)$ (we set $W_i(r) = V_i f_i(r)$, $i = 0, 1$), and the distance where $W_i(r)$ falls from 90% to 10% of the central value $W_i(0)$, respectively, are just given

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by $R_{1/2} = R$ and $t \simeq 4.4a$ and they are usually introduced in order to characterise the potential strength and shape, being closely related to the corresponding quantities for nuclear density.

When $\beta \neq 1$, the parameters R, a, β are not those most readily related to experimentally significant quantities. Furthermore, the parameters $R_{1/2}$ and t, given in this case by the expressions

$$\begin{aligned} R_{1/2} &= R + a \ln\{2^{1/\beta}[1+\exp(-R/a)]-1\} \\ t &= a \ln\left(\frac{10^{1/\beta}[1+\exp(-R/a)]-1}{(\frac{10}{9})^{1/\beta}[1+\exp(-R/a)]-1}\right) \end{aligned} \tag{1.3}$$

do not also seem appropriate quantities to characterise the potential strength and shape, even though they only show a small variation when β varies due, chiefly, to their being punctual functionals of $W_i(r)$.

Indeed, a suitable characterisation of the real and imaginary parts of optical-model potential wells should take into account all elements of their radial distribution (this also holds when $\beta = 1$) with a suitable weight function that changes continuously when $W_i(r)$ varies (an appropriate choice for it when $W_i(r)$ is normalised to unity is [11] $-\mathrm{d}W_i(r)/\mathrm{d}r$) and such a characterisation must, necessarily, be of an integral type. Thus a discussion of the geometrical properties of optical-model potential wells is better carried out, at least when a $(\mathrm{WS})^\beta$ parametrisation represents an adequate description of the physical problem under consideration, in terms of quantities such as the central radius, the equivalent sharp radius, the equivalent RMS (root-mean square) radius, the surface width, etc (cf [12], § 2.4), that are defined in terms of linear moments of the derivative of the normalised potential well $W_i(r) = V_i f_i(r)$.

It is therefore easy to see from the above considerations and from the analytic structure of the form factors $f_i(r)$ (see equation (1.2)) that the functions $\mathscr{D}_{n,\beta}(a, R)$ defined by the integral

$$\mathscr{D}_{n,\beta}(a, R) = \int_0^\infty \frac{r^n}{\{1+\exp[(r-R)/a]\}^\beta}\,\mathrm{d}r \tag{1.4}$$

are involved in optical-model studies.

On the other hand, special types of such functions also appear in nuclear physics in relation to some developments of the energy density formalism [13]. In fact, if the energy (the same holds for any other quantity as long as it can be expressed as a function of $\rho(r)$ and its derivatives) of a nuclear system with a leptodermous saturating distribution function $\rho(r)$, i.e. such that $\lim_{r\to 0}\rho(r) = \rho_0$, $\lim_{r\to\infty}\rho(r) = 0$, can be expressed by means of a function of $\rho(r)$ and its derivatives, then it is possible to separate volume, surface and higher effects by expanding this energy in powers of the ratio of the surface thickness to the radius of the system.

Now if $\rho(r)$ has a Fermi shape such an expression for the energy will involve integrals of the type given in equation (1.4).

It follows from the above considerations that it will be of interest to set up suitable analytic expressions in order to determine the analytic structure of the functions $\mathscr{D}_{n,\beta}(a, R)$, above all, when the parameter $x = R/a$ assumes large values.

It is therefore easy to understand why the first steps towards the attainment of the analytic structure of the functions $\mathscr{D}_{n,\beta}(a, R)$ aimed to derive an asymptotic expansion for them when $x = R/a$ is large, generalising a Sommerfeld expansion [14] in this form, valid when $\beta = 1$. To this end Krivine and Treiner [13, 14] expanded $\mathscr{D}_{n,\beta}(a, R)$

as follows:

$$\mathscr{D}_{n,\beta}(a, R)=\frac{R^{n+1}}{n+1}+P_{n,\beta}(x)+W_{n,\beta}(x) \qquad x=R/a$$

where $P_{n,\beta}(x)$ is a polynomial of order n in x, whose coefficients are given in terms of an integral (see equation (1.6)), and $W_{n,\beta}(x)$ is a function that goes to zero faster than any power of x^{-1} as $x\to\infty$.

It turns out, as will be shown in § 2, that the actual expansion for $\mathscr{D}_{n,\beta}(a, R)$ is actually of the form

$$\mathscr{D}_{n,\beta}(a, R)=\frac{R^{n+1}}{n+1}+n!\,a^{n+1}\sum_{\nu=0}^{n}\frac{x^{n-\nu}}{(n-\nu)!}\lambda_\nu(\beta)$$
$$+n!\,a^{n+1}(-1)^n\sum_{\nu=1}^{\infty}\frac{\Gamma(\beta+\nu)}{\Gamma(\beta)}\frac{(-1)^{\nu-1}}{\nu!}\frac{\mathrm{e}^{-\nu x}}{\nu^{n+1}} \qquad x=R/a \tag{1.5}$$

where the coefficients $\lambda_\nu(\beta)$ are given by the integral

$$\lambda_\nu(\beta)=\frac{(-1)^\nu}{\nu!}\int_0^\infty t^\nu\left(\frac{1+(-1)^\nu\,\mathrm{e}^{-\beta t}}{(1+\mathrm{e}^{-t})^\beta}-1\right)\mathrm{d}t. \tag{1.6}$$

From a physical point of view, the importance of such an expansion for optical-model potential wells of a leptodermous type, like those considered above, lies in the fact that it can be taken as a basis for expanding physical parameters (of relevance in the geometrical characterisation of optical potential wells and nuclear density distribution functions, as well as in the interpretation of experimental data) such as the central radius C or the RMS radius Q (see § 3) in powers of the skin coefficient $\alpha=b/\eta$, as follows:

$$C=\eta(1-\alpha^2-\ldots)$$
$$Q=\eta(1+\tfrac{5}{2}\alpha^2+\ldots) \qquad \alpha=b/\eta$$

where η, b stand for the sharp radius (the most significant quantity in such a characterisation) and the surface width respectively. Such an expansion, together with the relationships (3.31) and (3.32), allows us to compare and correlate experimental data with those obtained theoretically and also to inter-relate some properties of the alluded potentials with the corresponding ones of their nuclear density distributions for medium-mass and heavy nuclei.

On the other hand, as another example of the application of expression (2.5) (to which (1.5) reduces when R/a takes intermediate and large values; cf § 2), in the above-mentioned work of Treiner *et al* [13] (see, for example, § 3.4), use is made of such an expression in connection with the expansion of the so-called scaling and constrained incompressibility nuclear moduli K_A^s and K_A^c in powers of $A^{-1/3}$ (A stands for the nuclear mass number).

In fact, by expressing these moduli as functions of the nuclear density distribution and its derivatives and assuming a Fermi shape, special types of functions $\mathscr{D}_{n,\beta}(a, R)$ appear in these expressions.

The required expansions in powers of $A^{-1/3}$ are obtained in [13] by use of both expression (2.5) and the approximation $a/R=(a/\mu_0)A^{-1/3}$ with a and μ_0 constants.

It should be pointed out in considering this aspect that such a procedure is erroneous since, as will be seen in § 3, the actual expansion is obtained by use of expression (2.5) and Elton's [15] approximation (see equation (3.12)).

The material of this paper is organised as follows.

In § 2 the expansion (1.5) for $\mathscr{D}_{n,\beta}(a, R)$, as well as the polynomial to which it reduces when $e^{-R/a} \ll 1$, is obtained. On the other hand, several functional expansions are derived for the coefficients $\lambda_\nu(\beta)$ appearing in the above-mentioned expansions for $\mathscr{D}_{n,\beta}(a, R)$, and their respective convergence properties are analysed. In addition, a recursion relation that makes the numerical computation of these coefficients easier is also established.

In § 3, in order to gain some insight into the physical interpretation of (1.5) and the coefficients $\lambda_\nu(\beta)$ the approximate polynomial for $\mathscr{D}_{n,\beta}(a, R)$ is applied for the case of a leptodermous distribution function of type $(\mathrm{ws})^\beta$ to the study of the A dependence of certain quantities of relevance in the characterisation of geometrical properties of density distribution functions and optical-model potential wells corresponding to medium-mass and heavy nuclei.

2. Series expansions for $\mathscr{D}_{n,\beta}(a, R)$ and $\lambda_\nu(\beta)$

In this section we propose to obtain functional series expansions, in terms of which $\mathscr{D}_{n,\beta}(a, R)$ and the coefficients $\lambda_\nu(\beta)$, defined in the preceding section, can be expressed.

As a consequence of the physical interpretation of $\mathscr{D}_{n,\beta}(a, R)$ and $\lambda_\nu(\beta)$, let us assume throughout this paper that the parameters R, a and β in the expressions (1.4) and (1.6) take only positive, real values. Moreover, it should be emphasised that we are primarily interested in those values of R and a for which the parameter $x = R/a$ takes intermediate and large values.

In order to achieve our purpose, we note that equation (1.4), after some easy transformations, can be brought to the form

$$\mathscr{D}_{n,\beta}(a, R) = \frac{R^{n+1}}{n+1} + a^{n+1}\int_0^\infty \left[\frac{(x+t)^n}{(1+e^t)^\beta} - \left(1 - \frac{1}{(1+e^{-t})^\beta}\right)(x-t)^n\right] dt$$
$$+ a^{n+1}\int_x^\infty \left(1 - \frac{1}{(1+e^{-t})^\beta}\right)(x-t)^n\, dt \qquad x = R/a. \tag{2.1}$$

Therefore, by use of the binomial series

$$\frac{1}{(1+e^{-x}\,e^{-t})^\beta} = \sum_{\nu=0}^{\infty} \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \frac{(-1)^\nu}{\nu!} e^{-\nu x}\, e^{-\nu t} \tag{2.2}$$

to expand the integrand of the last integral in equation (2.1), one has

$$\mathscr{D}_{n,\beta}(a, R) = \frac{R^{n+1}}{n+1} + a^{n+1}\int_0^\infty \left[\frac{(x+t)^n}{(1+e^t)^\beta} - \left(1 - \frac{1}{(1+e^{-t})^\beta}\right)(x-t)^n\right] dt$$
$$+ a^{n+1} n!(-1)^n \sum_{\nu=1}^{\infty} \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \frac{(-1)^{\nu-1}}{\nu!} \frac{e^{-\nu x}}{\nu^{n+1}} \qquad x = R/a. \tag{2.3}$$

Finally, by use of the binomial expansion for $(x \pm t)^n$, equation (2.3) can be put into the form

$$\mathscr{D}_{n,\beta}(a, R) = \frac{R^{n+1}}{n+1} + a^{n+1}\sum_{\nu=0}^{n}\binom{n}{\nu} x^{n-\nu}(-1)^\nu \int_0^\infty \left(\frac{1+(-1)^\nu e^{-\beta t}}{(1+e^{-t})^\beta} - 1\right) t^\nu\, dt$$
$$+ a^{n+1} n!(-1)^n \sum_{\nu=1}^{\infty} \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \frac{(-1)^{\nu-1}}{\nu!} \frac{1}{\nu^{n+1}} \qquad x = R/a \tag{2.4}$$

which can also be rewritten as shown in equation (1.5).

It should be pointed out that an expression similar to equation (1.5) has been obtained by Srivastava [16] (see his equations (11) and (13)). However, Srivastava's expression is marred by an error, as the factor $(-1)^n$ in the last term does not appear in it.

If $e^{-R/a}$ is small, as is the case for a leptodermous distribution function, then we can neglect the last term in equation (1.5) and $\mathscr{D}_{n,\beta}(a, R)$ can be approximated as follows:

$$\mathscr{D}_{n,\beta}(a, R) \simeq \frac{R^{n+1}}{n+1}\left(1+(n+1)! \sum_{\nu=0}^{n} \frac{\lambda_\nu(\beta)}{(n-\nu)!}(a/R)^{\nu+1}\right) \tag{2.5}$$

where the coefficients $\lambda_\nu(\beta)$ are given by equation (1.6).

Notice that this approximation will be less appropriate for those nuclei with a strong asphericity, due to the fact that the transition regions of their corresponding distribution functions are relatively wider.

When $e^{-R/a}$ is not small, as is the case for light nuclei, the error due to approximating $\mathscr{D}_{n,\beta}(a, R)$ by the expression (2.5) becomes large and it will therefore be inappropriate to represent these functions under these circumstances.

On the other hand, if equation (2.5) or (1.5) is to be useful, then suitable expansions should be found in order to express the coefficients $\lambda_\nu(\beta)$. One such expansion can be derived at once by substituting the binomial expansion (2.2) (with $x=0$) for $(1+e^{-t})^{-\beta}$ in equation (1.6). In fact, doing so, one has after some easy calculations the following expression for $\lambda_\nu(\beta)$:

$$\lambda_\nu(\beta)=\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{\Gamma(\beta)}\frac{(-1)^k}{k!}\frac{1}{(k+\beta)^{\nu+1}}+(-1)^\nu\sum_{k=1}^{\infty}\frac{\Gamma(\beta+k)}{\Gamma(\beta)}\frac{(-1)^k}{k!\,k^{\nu+1}}. \tag{2.6}$$

Notice that if $\beta=1$, then $\lambda_\nu(\beta)$ may be expressed [17] in terms of the Riemann zeta function as follows:

$$\lambda_\nu(1)=[1-(-1)^\nu](1-2^{-\nu})\zeta(\nu+1). \tag{2.7}$$

Similarly, $\mathscr{D}_{n,\beta}(a, R)$ may be expressed in this case in terms of the Fermi–Dirac functions $\mathscr{F}_n(R/a)$ as follows:

$$\mathscr{D}_{n,\beta}(a, R)=a^{n+1}n!\,\mathscr{F}_n(R/a).$$

It is easy to demonstrate that the series expansions in equation (2.6) converge absolutely and uniformly only when $\nu+1>\beta$ and diverge (and therefore are unsuitable for representing the coefficients $\lambda_\nu(\beta)$) when $\nu+1<\beta$. On the other hand, for $\beta=1$ and $\nu=0$ the convergence is only conditional.

At the same time, we also see that, except for large values of $\nu+1-\beta$, such expansions do not have nice convergence properties. Hence it would be an advantage to derive an alternative series expansion endowed with better convergence properties than the one in equation (2.6).

This can be accomplished by substituting the following expansions

$$\frac{1}{(1+e^{-t})^\beta}=\frac{1}{2^\beta}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{\Gamma(\beta)}\frac{1}{k!}\frac{1}{2^k}(1-e^{-t})^k \tag{2.8a}$$

$$\left(\frac{1}{(1+e^{-t})^\beta}-1\right)=\frac{1}{2^\beta}\sum_{k=1}^{\infty}\frac{\Gamma(\beta+k)}{\Gamma(\beta)}\frac{1}{k!}\frac{1}{2^k}[(1-e^{-t})^k-1] \tag{2.8b}$$

valid for $\forall t\in[0,\infty]$, in the integral of equation (1.6), instead of using the binomial expansion considered above for $(1+e^{-t})^{-\beta}$.

Doing so, we obtain after some easy transformations the following alternative expansion for $\lambda_\nu(\beta)$:

$$\lambda_\nu(\beta)=\frac{\beta}{2^\beta}\sum_{s=0}^{\infty}\frac{\Gamma(1+\beta+s)}{\Gamma(1+\beta)}\frac{1}{s!}\frac{1}{2^s}\left(\frac{\sigma_s^\nu(\beta)}{s+\beta}-(-1)^\nu\frac{\sigma_s^{\nu+1}(1)}{2}\right) \tag{2.9}$$

where the coefficients $\sigma_s^\nu(\beta)$ may be defined by the equivalent relationships

$$\sigma_s^\nu(\beta)=\sum_{k=0}^{s}\binom{s}{k}\frac{(-1)^k}{(\beta+k)^{\nu+1}} \tag{2.10a}$$

$$\sigma_s^\nu(\beta)=\frac{1}{\nu!}\int_0^\infty t^\nu(1-e^{-t})^s\,e^{-\beta t}\,dt \qquad s=0,1,\ldots. \tag{2.10b}$$

These coefficients satisfy several recursion relations. The most important one, from a numerical point of view, is

$$\begin{aligned}
&\sigma_s^\nu(\beta)=\frac{s}{s+\beta}\sigma_{s-1}^\nu(\beta)+\frac{1}{s+\beta}\sigma_s^{\nu+1}(\beta) && s,\ \nu\geqslant 1\\
&\sigma_s^0(\beta)=s!\frac{\Gamma(\beta)}{\Gamma(1+\beta+s)} && \nu=0,\ s\geqslant 0\\
&\sigma_0(\beta)=1/\beta^{\nu+1} && \nu>0,\ s=0.
\end{aligned} \tag{2.11}$$

This recursion is a rapid and powerful tool for evaluating $\sigma_s^\nu(\beta)$ and, consequently, it also allows us a rapid and precise computation of the coefficients $\lambda_\nu(\beta)$ via expansion (2.9).

On the other hand, by writing equation (2.10*b*) as

$$\sigma_s^\nu(\beta)=\frac{(\ln(s))^\nu}{s^\beta}\frac{1}{\nu!}\int_0^s\left(1-\frac{\ln(t)}{\ln(s)}\right)^\nu\left(1-\frac{t}{s}\right)^s t^{\beta-1}\,dt \tag{2.12}$$

it is easy to see that the following expression holds:

$$\lim_{s\to\infty}\frac{\sigma_s^\nu(\beta)}{(\Gamma(\beta)/\nu!)(\ln^\nu(s)/s^\beta)}=1. \tag{2.13}$$

Therefore it follows from this latter relationship that the expansion in equation (2.9) converges absolutely and uniformly $\forall\beta\Rightarrow\beta\in(0,\infty)$ for every finite value of ν. Moreover, if $R_N(\beta,\nu)$ denotes the N-term remainder of this expansion, then it can be demonstrated that

$$|R_N(\beta,\nu)|\leqslant\beta 2^{-N}\left|\frac{\sigma_{N+1}^\nu(\beta)}{N+1+\beta}-\frac{\sigma_{N+1}^{\nu+1}(1)}{2}\right|\prod_{m=0}^{N}\left(1+\frac{\beta}{1+m}\right). \tag{2.14}$$

Hence the expansion in equation (2.9) has nice convergence properties in a wide interval of β. In particular, for all those values of β in the interval $[1,2]$, in which we are chiefly interested (cf the recursion relation (2.22)), these convergence properties are excellent. It can also be shown that, after some easy calculations, this expansion can be transformed as follows:

$$\lambda_\nu(\beta)=\frac{1}{\beta^{\nu+1}}-2^{-\beta}\beta\sum_{s=0}^{\infty}\frac{\Gamma(1+\beta+s)}{\Gamma(1+\beta)}\frac{1}{s!}\frac{1}{2^s}\left(\frac{\mathscr{H}_s^0(\beta)}{\beta+s}+(-1)^\nu\frac{\sigma_s^{\nu+1}(1)}{2}\right) \tag{2.15}$$

where the coefficients $\mathscr{H}_s^\nu(\beta)$, which may be expressed in terms of the coefficients $\sigma_s^\nu(\beta)$ by

$$\mathscr{H}_s^\nu(\beta) = (1/\beta^{\nu+1}) - \sigma_s^\nu(\beta) \tag{2.16}$$

are given by the integral

$$\mathscr{H}_s^\nu(\beta) = \frac{1}{\nu!}\int_0^\infty t^\nu[1-(1-e^{-t})^s]\,e^{-\beta t}\,dt. \tag{2.17}$$

Now, equations (2.15) and (2.17) clearly reveal that $\lambda_\nu(\beta)$, considered as a function of β, has a pole of order $\nu+1$ at $\beta=0$.

To conclude this section, let us derive an interesting recursion relation satisfied by the coefficients $\lambda_\nu(\beta)$. In fact, to this end we first note that, after some easy transformations, equation (1.6) can be brought to the form

$$\lambda_\nu(\beta) = \frac{1}{\nu!}\left[\int_0^1 \frac{\ln^\nu(t)}{t}\left(\frac{1}{(1+t)^\beta}-1\right)dt + \int_1^\infty \frac{\ln^\nu(t)}{t(1+t)^\beta}\,dt\right] \tag{2.18}$$

and by partial integration of this equation, to the form

$$\lambda_\nu(\beta) = \frac{\beta}{(\nu+1)!}\int_0^\infty \frac{\ln^{\nu+1}(t)}{(1+t)^{\beta+1}}\,dt. \tag{2.19}$$

Now, by inserting the relationship

$$\frac{1}{t(1+t)^{\beta+1}} = \frac{1}{t(1+t)^\beta} - \frac{1}{(1+t)^{1+\beta}} \tag{2.20}$$

into equation (2.18) (with β replaced by $\beta+1$), we obtain

$$\lambda_\nu(\beta+1) = \lambda_\nu(\beta) - \frac{1}{\nu!}\int_0^\infty \frac{\ln^\nu(t)}{(1+t)^{1+\beta}}\,dt. \tag{2.21}$$

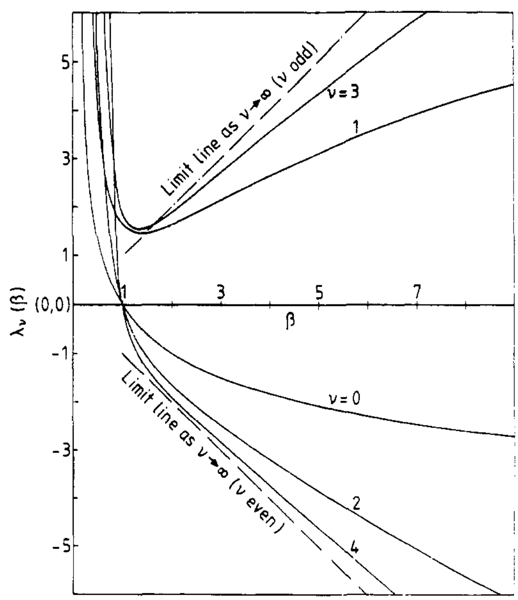

Figure 1. Variation of the coefficients $\lambda_\nu(\beta)$ with β in the interval $[0, 9]$ for $\nu = 0, 1, \ldots, 4$.

Finally, taking into account equation (2.19), we see at once from equation (2.21) that the coefficients $\lambda_\nu(\beta)$ fulfil the recursion relation

$$\lambda_\nu(\beta+1)=\lambda_\nu(\beta)-(1/\beta)\lambda_{\nu-1}(\beta) \qquad \lambda_{-1}(\beta)=1 \qquad \nu=0,1,\ldots. \tag{2.22}$$

It should be emphasised that, by successive applications of this recursion, the problem of the evaluation of the coefficients $\lambda_\nu(\beta)$ for $\beta>2$ can be reduced to the problem of their evaluation for $\beta\in[1,2]$.

The variation of the first coefficients $\lambda_\nu(\beta), \forall\beta\in[0,9]$ has been displayed in figure 1.

3. Applications

In this section, by assuming a $(\mathrm{ws})^\beta$ parametrisation for the nuclear density distribution functions and optical-model potential wells corresponding to medium-mass and heavy nuclei, we shall apply the relationship (2.5) (found in the preceding section to represent the functions $\mathscr{D}_{n,\beta}(a, R)$ when $\mathrm{e}^{-R/a}\ll 1$) to the study of the A dependence of certain quantities of great experimental interest in the characterisation of the shape and extent of such density distributions and potential wells.

As stated in the introduction, the parameters R, a and β (see equation (3.1)) are not, in this case, those most readily related to experimentally significant quantities. Certain functions of these parameters are more relevant and, therefore, it is interesting to obtain them explicitly. At the same time, it will allow us to gain some insight into the physical interpretation of the coefficients $\lambda_\nu(\beta)$. It should be borne in mind, however, that the present generalisation of the ws parametrisation may itself not be an adequate description of the physical problem under consideration in some cases.

Let us thus assume that the nuclear matter in a spherical nucleus of mass number A may be represented by a leptodermous density distribution function of the form

$$\rho(r)=\frac{\rho_0}{\{1+\exp[(r-R)/a]\}^\beta} \qquad \mathrm{e}^{-R/a}\ll 1 \tag{3.1}$$

where ρ_0, r, R and β are positive real parameters. Then, from the normalisation condition

$$A=4\pi\int_0^\infty \rho(r)r^2\,\mathrm{d}r \tag{3.2}$$

and from equations (1.4) and (3.1), we obtain

$$A=4\pi\rho_0\mathscr{D}_{2,\beta}(a, R). \tag{3.3}$$

Now if as supposed, $\mathrm{e}^{-R/a}\ll 1$, by making use of relationship (2.5) it is possible to transform this equation as follows:

$$\begin{aligned}(a/\mu_0)A^{-1/3}\simeq{}&(a/R)-\lambda_0(\beta)(a/R)^2-2(\lambda_1(\beta)-\lambda_0^2(\beta))(a/R)^3\\&-2(\lambda_2(\beta)-4\lambda_1(\beta)\lambda_0(\beta)+\tfrac{7}{3}\lambda_0^3(\beta))(a/R)^4\\&+4(2\lambda_2(\beta)\lambda_0(\beta)+2\lambda_1^2(\beta)-7\lambda_1(\beta)\lambda_0^2(\beta))(a/R)^5+\ldots\end{aligned} \tag{3.4}$$

where

$$\mu_0=(\tfrac{4}{3}\pi\rho_0)^{-1/3} \tag{3.5}$$

and the terms indicated by dots are $\mathrm{O}((a/R)^6)$.

Therefore by inverting expansion (3.4) one has for a/R

$$
\begin{aligned}
a/R = \frac{a}{\mu_0} A^{-1/3}\bigg(1 &+ \lambda_0(\beta)\frac{a}{\mu_0}A^{-1/3} + 2\lambda_1(\beta)\frac{a^2}{\mu_0^2}A^{-2/3} \\
&+ (2\lambda_2(\beta) + 2\lambda_1(\beta)\lambda_0(\beta) - \tfrac{1}{3}\lambda_0^3(\beta))\frac{a^3}{\mu_0^3}A^{-1} \\
&+ (4\lambda_2(\beta)\lambda_0(\beta) + 4\lambda_1^2(\beta) - 2\lambda_1(\beta)\lambda_0^2(\beta) + \tfrac{1}{3}\lambda_0^4(\beta))\frac{a^4}{\mu_0^4}A^{-4/3} \\
&+ 4(\lambda_2(\beta)\lambda_1(\beta) - \lambda_1^2(\beta)\lambda_0(\beta) + \tfrac{1}{3}\lambda_1(\beta)\lambda_0^3(\beta))\frac{a^5}{\mu_0^5}A^{-5/3} + \ldots\bigg)
\end{aligned}
\tag{3.6}
$$

and also

$$
\begin{aligned}
R \simeq \mu_0 A^{1/3}\bigg(1 &- \lambda_0(\beta)\frac{a}{\mu_0}A^{-1/3} - (2\lambda_1(\beta) - \lambda_0^2(\beta))\frac{a^2}{\mu_0^2}A^{-2/3} \\
&- 2(\lambda_2(\beta) - \lambda_1(\beta)\lambda_0(\beta) + \tfrac{1}{3}\lambda_0^2(\beta))\frac{a^3}{\mu_0^3}A^{-1} \\
&- 2(2\lambda_2(\beta)\lambda_1(\beta) - \lambda_2(\beta)\lambda_0^2(\beta) - 2\lambda_1^2(\beta)\lambda_0(\beta) \\
&+ \tfrac{5}{3}\lambda_1(\beta)\lambda_0^3(\beta) - \tfrac{1}{3}\lambda_0^5(\beta))\frac{a^5}{\mu_0^5}A^{-5/3} + \ldots\bigg).
\end{aligned}
\tag{3.7}
$$

It should be noticed that in the approximation under consideration the half-value radius $R_{1/2}$ and the surface thickness t are given by the relationships (see equations (1.3))

$$
\begin{aligned}
R_{1/2} &\simeq R + a\ln(2^{1/\beta} - 1) \\
t &\simeq a\ln\left(\frac{10^{1/\beta} - 1}{(\frac{10}{9})^{1/\beta} - 1}\right).
\end{aligned}
\tag{3.8}
$$

The integral counterparts of the punctual quantities $R_{1/2}$ and t are the central radius C and the surface width b, respectively, which are defined in terms of the surface distribution

$$
g(r) = -\frac{[1+\exp(-R/a)]^\beta}{\rho_0}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \tag{3.9}
$$

in the following way [11, 12]:

$$
C = \int_0^\infty g(r) r\,\mathrm{d}r \tag{3.10}
$$

$$
b^2 = \int_0^\infty g(r)(r - C)^2\,\mathrm{d}r. \tag{3.11}
$$

Hence from equations (3.9)-(3.11), and (1.4), one has

$$
C = [1+\exp(-R/a)]^\beta \mathscr{D}_{0,\beta}(a, r) \tag{3.12}
$$

and

$$
b^2 = 2[1+\exp(-R/a)]^\beta \mathscr{D}_{1,\beta}(a, R) - [1+\exp(-R/a)]^{2\beta}\mathscr{D}_{0,\beta}^2(a, R). \tag{3.13}
$$

Therefore in the approximation under consideration it follows from equation (3.12) and the relationships (2.5) and (3.7) for C, and from equation (3.13) and these relationships for b, that

$$C=\mu_0 A^{1/3}\Bigg(1-(2\lambda_1(\beta)-\lambda_0^2(\beta))\frac{a^2}{\mu_0^2}A^{-2/3}-2(\lambda_2(\beta)-\lambda_1(\beta)\lambda_0(\beta)+\tfrac{1}{3}\lambda_0^3(\beta))\frac{a^3}{\mu_0^3}A^{-1}$$
$$-2(2\lambda_2(\beta)\lambda_1(\beta)-\lambda_2(\beta)\lambda_0^2(\beta)-2\lambda_1^2(\beta)\lambda_0(\beta)$$
$$+\tfrac{5}{3}\lambda_1(\beta)\lambda_0^3(\beta)-\tfrac{1}{3}\lambda_0^5(\beta))\frac{a^5}{\mu_0^5}A^{-5/3}+\dots\Bigg) \tag{3.14}$$

and

$$b\simeq(2\lambda_1(\beta)-\lambda_0^2(\beta))^{1/2}a. \tag{3.15}$$

(It should be noticed that b is independent of the mass number A in this approximation, i.e. a is a constant. This result can only be valid as an average since b must obviously be affected by shell structure and nuclear deformations.)

The next quantities of interest are the *equivalent sharp radius* η and the RMS radius Q, respectively defined by the expressions

$$\tfrac{1}{3}\eta^3=\frac{[1-\exp(-R/a)]^\beta}{\rho_0}\int_0^\infty \rho(r)r^2\,\mathrm{d}r \tag{3.16}$$

and

$$Q^2=\frac{5}{3}\frac{\int_0^\infty \rho(r)r^4\,\mathrm{d}r}{\int_0^\infty \rho(r)r^2\,\mathrm{d}r}. \tag{3.17}$$

Therefore, taking into account equation (3.1) and (1.4), we have

$$\tfrac{1}{3}\eta^3=[1+\exp(-R/a)]^\beta\mathscr{D}_{2,\beta}(a,R) \tag{3.18}$$

and

$$Q^2=\frac{5}{3}\frac{\mathscr{D}_{4,\beta}(a,R)}{\mathscr{D}_{2,\beta}(a,R)}. \tag{3.19}$$

Hence in the approximation under consideration it follows from equations (3.18) and (3.3) for η, and from equations (3.19) and (3.3) and the expressions (2.5) and (3.7) for Q, that

$$\eta\simeq\mu_0 A^{1/3} \tag{3.20}$$

and

$$Q\simeq\mu_0 A^{1/3}\Bigg(1+\tfrac{5}{2}(2\lambda_1(\beta)-\lambda_0^2(\beta))\frac{a^2}{\mu_0^2}A^{-2/3}$$
$$+25(\lambda_2(\beta)-\lambda_1(\beta)\lambda_0(\beta)+\tfrac{1}{3}\lambda_0^2(\beta))\frac{a^3}{\mu_0^3}A^{-1}$$
$$+\tfrac{15}{2}(8\lambda_3(\beta)-8\lambda_2(\beta)\lambda_0(\beta)+22\lambda_1(\beta)\lambda_0^2(\beta)-7\lambda_1^2(\beta)-15\lambda_0^4(\beta))\frac{a^4}{\mu_0^4}A^{-4/3}$$
$$+(60\lambda_4(\beta)-60\lambda_3(\beta)\lambda_0(\beta)-275\lambda_2(\beta)\lambda_1(\beta)+\tfrac{335}{2}\lambda_2(\beta)\lambda_0^2(\beta)$$
$$+275\lambda_1^2(\beta)\lambda_0(\beta)-\tfrac{295}{2}\lambda_1(\beta)\lambda_0^3(\beta)+\tfrac{287}{6}\lambda_0^5(\beta))\frac{a^5}{\mu_0^5}A^{-5/3}+\dots\Bigg). \tag{3.21}$$

We see from equations (3.14), (3.20) and (3.21) that the quantity of greatest interest is η. By rewriting its defining expression (3.16) as

$$0=4\pi\int_0^\infty\left(\frac{[1+\exp(-R/a)]^\beta}{\rho_0}\rho(r)-\theta(\eta-r)\right)r^2\,\mathrm{d}r$$

where $\theta(\eta - r)$ stands for the step function, it follows at once that η represents the radius of a uniform sharp distribution, having the same bulk value and the same volume integral as the distribution $(1/\rho_0)[1+\exp(-R/a)]^\beta\rho(r)$.

The quantities $R_{1/2}$ and C are often used to characterise nuclear densities and optical-model potential wells. C and b are properties of nuclear distributions that can be determined by electron scattering experiments [18]. Finally, Q is of interest because it is a property of nuclear distributions that can be directly derived from μ-meson atom experiments [19].

If $\beta = 1$, i.e. in the case of a Fermi distribution, expressions (3.6), (3.7), (3.19) and (3.21) reduce (see equation (2.7)) to

$$(a/R) \simeq (a/\mu_0)A^{-1/3}\left(1+\frac{1}{3}\frac{\pi^2a^2}{\mu_0^2}A^{-2/3}+\frac{1}{9}\frac{\pi^4a^4}{\mu_0^4}A^{-4/3}+O(A^{-2})\right) \tag{3.22}$$

$$R = C \simeq \mu_0 A^{1/3}\left(1-\frac{1}{3}\frac{\pi^2a^2}{\mu_0^2}A^{-2/3}+O(A^{-2})\right) \tag{3.23}$$

$$Q = \mu_0 A^{1/3}\left(1+\frac{5}{6}\frac{\pi^2a^2}{\mu_0^2}A^{-2/3}-\frac{7}{24}\frac{\pi^4a^4}{\mu_0^4}A^{-4/3}+O(A^{-2})\right). \tag{3.24}$$

Expressions (3.23) and (3.24) are just Elton's formulae (6) and (7) [15]. Note, however, that his expression (7), corresponding to our Q, is incorrect since relationship (A4) in the appendix of his paper, from which (7) is derived, is unfortunately marred by an error.

On the other hand, by taking into account equation (3.20) and the skin coefficient b/η given by the expression

$$(b/\eta) \simeq (2\lambda_1(\beta)-\lambda_0^2(\beta))^{1/2}(a/\mu_0)A^{-1/3} \tag{3.25}$$

we can rewrite the expressions (3.14) and (3.21) in terms of η and b/η as follows:

$$C \simeq \eta[1-(b/\eta)^2-\tfrac{1}{3}\gamma_3(\beta)(b/\eta)^3-\tfrac{1}{3}\gamma_3(\beta)(b/\eta)^5+\ldots] \tag{3.26}$$

$$Q \simeq \eta[1+\tfrac{5}{2}(b/\eta)^2+\tfrac{25}{6}\gamma_3(\beta)(b/\eta)^3+\tfrac{5}{2}(\gamma_4(\beta)-\tfrac{21}{4})(b/\eta)^4 + \tfrac{1}{2}(\gamma_5(\beta)-\tfrac{275}{6}\gamma_3(\beta))(b/\eta)^5+\ldots] \tag{3.27}$$

where the coefficients $\gamma_3(\beta)$, $\gamma_4(\beta)$ and $\gamma_5(\beta)$ are respectively given by the expressions

$$\gamma_3(\beta) = 2\frac{3\lambda_2(\beta)-3\lambda_1(\beta)\lambda_0(\beta)+\lambda_0^3(\beta)}{(2\lambda_1(\beta)-\lambda_0^2(\beta))^{3/2}} \tag{3.28}$$

$$\gamma_4(\beta) = 3\frac{8\lambda_3(\beta)-8\lambda_2(\beta)\lambda_0(\beta)+4\lambda_1(\beta)\lambda_0^2(\beta)-\lambda_0^4(\beta)}{(2\lambda_1(\beta)-\lambda_0^2(\beta))^2} \tag{3.29}$$

$$\gamma_5(\beta) = 4\frac{30\lambda_4(\beta)-30\lambda_3(\beta)\lambda_0(\beta)+15\lambda_2(\beta)\lambda_0^2(\beta)-5\lambda_1(\beta)\lambda_0^3(\beta)+\lambda_0^5(\beta)}{(2\lambda_1(\beta)-\lambda_0^2(\beta))^{5/2}}. \tag{3.30}$$

By comparing (3.26) and (3.27) with Süssmann's [11] expressions (5.12) and (5.13), we see at once that $\gamma_3(\beta)$, $\gamma_4(\beta)$ and $\gamma_5(\beta)$ are just the flair, crookedness and fifth shape coefficient of the surface distribution (3.9). Therefore, the combination of the coefficients $\lambda_\nu(\beta)$ given by the expressions (3.28)-(3.30) provides us with additional information about the structure of the nuclear surface layer, i.e. with information different from position and size, which are obviously given by C and b, respectively.

At this point, it is also convenient to emphasise that by fitting sufficient precise experimental data for the ratio $C/A^{1/3}$ or $Q/A^{1/3}$ (obtained, for example, from electron scattering or μ-meson atom experiments carried out with medium-mass and heavy nuclei) to expressions (3.26) or (3.27), it might, in principle, be possible to obtain the value of the parameter $\gamma_3(\beta)$ and with it, through equation (3.28), to determine also the value of the exponent β in the nuclear distribution function $\rho(r)$.

The earlier considerations with respect to the geometrical relationships apply equally well to optical-model potential wells. However, as Myers [12] points out, the main difference with respect to the case of nuclear distribution functions lies in the fact that the corresponding equivalent sharp radius η is not now proportional to $A^{1/3}$, i.e. equation (3.20) is not valid, since the normalisation condition (3.2) does not apply in this case. Myers finds that equations (3.20) and (3.15) should be replaced, in the case of optical-model potential wells, by the following relationships:

$$\eta = 1.16A^{1/3} + 0.45 \tag{3.31}$$

$$b = 1.13 \tag{3.32}$$

where η, b are given in fm.

The corresponding expansions for C and Q in powers of $A^{-1/3}$ are now easily obtained by first deriving the expansion for the skin coefficient

$$b/\eta = 1.12A^{-1/3}(1 + 0.39A^{-1/3})^{-1} \tag{3.33}$$

and then substituting this expansion into the expressions (3.26) and (3.27).

Another interesting point to be emphasised is the puzzling situation that happens when one admits that quantities such as $R_{1/2}$, C or Q are strictly proportional to $A^{1/3}$. This erroneous approach is found in a number of places. For example, in the paper of Treiner *et al* [13], referred to earlier, by assuming a nuclear distribution function of Wood–Saxon type, it is shown that the so-called scaling and constrained incompressibility nuclear moduli K^{s}_A and K^{c}_A can be expressed in terms of the functions $\mathscr{D}_{2,\beta}(a, R)$ for different values of β. Then in order to obtain the expansion of these coefficients in powers of $A^{-1/3}$, use is made of the expression (2.5) and the approximation $R = \mu_0 A^{1/3}$, so that the surface, curvature, etc, contributions to K^{s}_A and K^{c}_A are incorrectly calculated. The correct procedure for obtaining the actual values of these contributions should involve the use of expressions (2.5) and (3.23).

On the other hand, it should be noticed that the nuclear distribution function (3.1) under the conditions considered for $\mathrm{e}^{-R/a}$ in the present section, may be rewritten (see equations (3.7), (3.14), (3.15), (3.25) and (3.26)) in terms of physically significant quantities, such as η and b, as follows:

$$\rho(r) \simeq \rho_0\left[1 + \exp\left(\frac{\delta(\beta)}{b}\left\{r - \eta\left[1 - \frac{\lambda_0(\beta)}{\delta(\beta)}\left(\frac{b}{\eta}\right) - \left(\frac{b}{\eta}\right)^2 - \frac{1}{3}\gamma_3(\beta)\left(\frac{b}{\eta}\right)^3 - \ldots\right]\right\}\right)\right]^{-\beta} \tag{3.34}$$

where

$$\delta(\beta) = (2\lambda_1(\beta) - \lambda_0^2(\beta))^{1/2}. \tag{3.35}$$

Note, at the same time, the $A^{1/3}$ dependence of $\rho(r)$ through η (see equation (3.20)).

Finally, to conclude this section, let us add some very brief comments in relation to the N, Z dependence of the equilibrium properties of nuclei.

As stated in a paper by Hilf and Wolff [20], in order to exhibit the gross trends of this dependence, the nuclear binding energy is described in macroscopic models as a balance of several driving and restoring forces, such as nuclear matter saturation, neutron excess, etc. Each of them can now be expressed in terms of quantitities of type η, b, $\gamma_3(\beta)$, $\gamma_4(\beta), \ldots$, corresponding to neutron and proton distribution functions (a functional form of type (3.1) can also be assumed for these distributions). In the droplet model of Myers and Swiatecki [21], the equilibrium is found by varying only one parameter, i.e. $R_{1/2}$. A more general model might be set up that eventually includes the variation of η, b and β. This question, however, deserves further study and will be discussed elsewhere.

4. Conclusions

Our main results in this paper can be summarised as follows.

(i) The analytic structure of the moments $\mathscr{D}_{n,\beta}(a, R)$ has been suitably established by obtaining the correct series expansion for them, when a, R and β take all real and positive values.

(ii) When $e^{-R/a} \ll 1$, this expansion reduces to a polynomial of order n in $x = R/a$. The analytic structure of its coefficients has also been established by deriving suitable converging series expansions for them. In addition, a recursion relation held by these coefficients, that makes their numerical computation easier, has also been found.

(iii) The A dependence of certain quantities of relevance in the geometrical characterisation of density distribution functions and optical-model potential wells for medium-mass and heavy nuclei has also been studied and, on the basis of a density distribution of the form $(\mathrm{ws})^{\beta}$ and the use of this polynomial, several interesting relationships have been established for them. When $\beta = 1$, they reduce to the well known formulae of Elton.

To conclude, we hope that the above mathematical results will be of help not only in optical-model studies and for the study of nuclei using the energy density formalism but also, by assuming a distribution of a generalised Fermi type, in other fields of physics, such as semiconductor band theory.

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